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Three-dimensional long-wave instability of unidirectional spatially periodic viscous flows

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The long-wave instability of unidirectional spatially periodic flows is investigated by means of asymptotic expansions. It is shown that the wavevector of the most dangerous disturbances is generally inclined to the direction of the basic stream. A new type of long-wave oscillatory instability is discovered, and a comparison with results of previous investigations is performed.

1. Introduction

The phenomenon of spontaneous generation of large-scale structures by small-scale flows is observed in different physical contexts (coherent structures in turbulent flows, atmospheric cyclones, etc.) The nature of this phenomenon is not yet fully clear. Kraichnan (1967, 1976) suggested that the large-scale structures are produced by an inverse cascade driven by the long-wavelength instability of small-scale flows. However, the details of this process are unknown. That is why the investigation of particular relatively simple cases may be useful for solving this intriguing problem.

In the 1950s, Kolmogorov proposed investigating the stability of the flow generated by an unidirectional spatially periodic force with a sinusoidal velocity profile, in order to understand the cascade processes in turbulent flows. The linear stability theory was developed by Meshalkin & Sinai (1961) who found that the Kolmogorov flow is unstable with respect to long-wavelength disturbances, unlike the flows in channels and boundary layers which are subject to short-wavelength instabilities. The nonlinear evolution of long-wavelength disturbances, which is governed by some modifications of the Cahn–Hilliard equation (Nepomnyashchy 1976; Sivashinsky 1985), mimics the spontaneous generation of large-scale structures by small-scale flows and the inverse cascade (She 1987).

It is generally accepted now that the investigation of long-wavelength instabilities of flows generated by an external force could help to understand the nature of the self-organization of flow into large-scale structures. Up to now, extensive investigation has been concentrated mainly on the Kolmogorov flow (see E & Shu 1993; Borue & Orszag 1996 and reference therein). Even the linear stability has been developed only for some simple particular classes of primary flows. The existence of a two-dimensional long-wave instability was established by Yudovich (1966) in the case of an arbitrary parallel spatially periodic flow

$$\boldsymbol{u} = (0, u_2(x_1)), \quad u_2(x_1 + L_1) = u_2(x_1). \tag{1.1}$$

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A generalization of the long-wave stability theory in the case of an unidirectional flow periodic in two coordinates

$$u = (0, u_2(x_1, x_3), 0), \quad u_2(x_1 + L_1, x_3) = u_2(x_1, x_3 + L_3) = u_2(x_1, x_3) \tag{1.2}$$

can be found in Shtilman & Sivashinsky (1986), Yakhot & Sivashinsky (1987) and Brutyan & Krapivsky (1991).

It should be noted that the abovementioned investigations were incomplete in the following sense: periodic boundary conditions in directions normal to the direction of the basic flow were postulated from the very beginning. However, there is a crucial difference between flows in a channel, where the disturbances certainly satisfy the same boundary conditions as the basic flow, and flows in an unbounded space generated by an external periodic force. In the latter case, the flow is not necessarily periodic, and the only physical restriction for disturbances is boundness. This means that the disturbances can be considered in the form of Floquet functions, e.g. in the case (1.1)

$$u' = w(x_1) \exp[i(\kappa_1 x_1 + \kappa_2 x_2) + St], \quad w(x_1 + L_1) = w(x_1). \tag{1.3}$$

Gotoh, Yamada & Mizushima (1983) were the first investigators who considered the stability of flows with respect to disturbances in the form (1.3). However, only in the remarkable paper of Dubrulle & Frisch (1991) was a really non-trivial result obtained: in the generic case, the wavevector of the most dangerous disturbance of the flow (1.1) is inclined to the direction of the basic flow.

In the present paper, we perform an investigation of the long-wave instability of unidirectional flows (1.2) periodic in two coordinates with respect to arbitrary infinitesimal bounded disturbances. We obtain an equation determining an explicit general expression for growth rates of long-wave disturbances, which makes it possible to determine the wavevector's direction for the most dangerous disturbance, to find the critical Reynolds number and to verify the results obtained by previous authors.

This paper is organized as follows. In §2 the derivation of the general dispersion relation is presented. In §§3 and 4 we consider some particular flows of the type (1.1) and (1.2), respectively. Section 5 contains some concluding remarks.

2. Stability of periodic unidirectional flows

The problem is governed by the system of Navier–Stokes equations

$$\left.\begin{aligned} &(\nabla \cdot u) = 0, \\ &\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla p + \nu \Delta u + f \end{aligned}\right\} \tag{2.1}$$

where the dimensionless parameter ν is the inverse of the Reynolds number: $\nu = 1/Re$.

We assume that three-dimensional incompressible motion is generated by a steady external body force, having a sole coordinate different from zero $f = (0, f_2, 0)$. The function $f_2 = f_2(x_1, x_3)$ is assumed to be periodic in two spatial coordinates: $f_2(x_1 + L_1, x_3) = f_2(x_1, x_3 + L_3) = f_2(x_1, x_3)$. We suppose that the mean value of the function f_2 over the periodicity cell is zero. We assume also that it is possible to represent the function f_2 by an infinite Fourier series

$$f_2(x_1, x_3) = \sum_{m,n=-\infty}^{\infty}{}' f_{mn} \exp\left\{2\pi i \left(\frac{x_1 m}{L_1} + \frac{x_3 n}{L_3}\right)\right\},$$

where the prime denotes exclusion of the term $(m, n) = (0, 0)$. Thus,

$$\langle f_2(x_1, x_3) \rangle = f_{00} = 0 \, , \tag{2.2}$$

where the averaging operation is defined by the formula

$$\langle \cdot \rangle = \frac{1}{L_1 L_3} \int_0^{L_1} \int_0^{L_3} \cdot \; \mathrm{d}x_1 \mathrm{d}x_3. \tag{2.3}$$

2.1. *Basic flow*

For any ν, the problem (2.1) has a particular solution describing a steady spatially periodic parallel flow $\{\boldsymbol{u}^{(0)} = (0, u_2, 0),\ p^{(0)} = P\}$, where $u_2 = u_2(x_1, x_3)$ is a periodic function with

$$u_2(x_1 + L_1, x_3) = u_2(x_1, x_3 + L_3) = u_2(x_1, x_3). \tag{2.4}$$

This solution satisfies the following reduced system of equations:

$$\partial_1 P = 0, \tag{2.5a}$$

$$\partial_2 P + \nu \Delta u_2 + f_2 = 0, \tag{2.5b}$$

$$\partial_3 P = 0 \tag{2.5c}$$

where $\partial_i = \partial / \partial x_i, i = 1{,}2{,}3$. Because of Galileo's principle, we can add the condition

$$\langle u_2 \rangle = 0. \tag{2.6}$$

Equations (2.5*a*) and (2.5*c*) lead to $P = P(x_2)$. Averaging equation (2.5*b*), we obtain $\langle \partial_2 P(x_2) \rangle = 0$, according to (2.2) and (2.6). Thus, we get

$$P = \text{const.} \tag{2.7}$$

Using the result (2.7), we obtain from (2.5*b*):

$$\nu \Delta u_2 + f_2 = 0. \tag{2.8}$$

Conditions (2.4), (2.6) and (2.8) define the function u_2 uniquely.

Let us define the operator Δ^{-1} on the class of the spatially periodic functions $g(x_1 + L_1, x_3) = g(x_1, x_3 + L_3) = g(x_1, x_3)$ with zero average value $\langle g(x_1, x_3) \rangle = 0$ by means of the formula

$$\Delta^{-1} g(x_1, x_3) = -\frac{1}{4\pi^2} \sum_{m,n=-\infty}^{\infty}{}' \frac{g_{mn}}{(m/L_1)^2 + (n/L_3)^2} \exp\left\{ 2\pi \mathrm{i} \left(\frac{x_1 m}{L_1} + \frac{x_3 n}{L_3} \right) \right\} \tag{2.9}$$

where g_{mn} are Fourier coefficients of the function $g(x_1, x_3)$ and the prime denotes exclusion of the term $(m, n) = (0, 0)$.

We can write $u_2 = -\nu^{-1} \Delta^{-1} f_2$. According to the definition (2.9), the mean value of any function $\Delta^{-1} f_2$ is zero for all f_2.

2.2. *Stability problem*

One can expect that the basic parallel flow $\boldsymbol{u}^{(0)}$ is unstable for sufficiently small values of ν. The instability will be analysed by means of the linear stability theory. Let us impose a small disturbance $\boldsymbol{u}' = (u_1', u_2', u_3'), p'$ on the basic solution by the following substitution:

$$\boldsymbol{u} \to \boldsymbol{u}^{(0)} + \boldsymbol{u}',$$

$$p \to p^{(0)} + p'.$$

Neglecting all the nonlinear terms we obtain the following system of equations:

$$\left.\begin{aligned}&\partial_1 u_1' + \partial_2 u_2' + \partial_3 u_3' = 0,\\&\partial_t u_1' + u_2\partial_2 u_1' = -\partial_1 p' + \nu\Delta u_1',\\&\partial_t u_2' + u_2\partial_2 u_2' + \partial_1 u_2 u_1' + \partial_3 u_2 u_3' = -\partial_2 p' + \nu\Delta u_2',\\&\partial_t u_3' + u_2\partial_2 u_3' = -\partial_3 p' + \nu\Delta u_3'\end{aligned}\right\} \tag{2.10}$$

where $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$, $\partial_t = \partial/\partial t$, $\partial_i = \partial/\partial x_i, i = 1,2,3$.

We shall seek a solution of the system (2.10) in the following form:

$$\begin{aligned}\boldsymbol{u}' &= \boldsymbol{w}'(x_1,x_2,x_3)\mathrm{e}^{St},\\ p' &= P'(x_1,x_2,x_3)\mathrm{e}^{St},\end{aligned}$$

where S is the growth rate, and the functions $\boldsymbol{w}', P'$ are bounded on R^3 including the points $x_j \to \infty, j = 1,2,3$.

Thus we obtain the following eigenvalue problem for the functions P' and $\boldsymbol{w}'$:

$$\left.\begin{aligned}&\nabla\cdot\boldsymbol{w}' = 0,\\&Sw_1' + u_2\partial_2 w_1' = -\partial_1 P' + \nu\Delta w_1',\\&Sw_2' + u_2\partial_2 w_2' + \partial_1 u_2 w_1' + \partial_3 u_2 w_3' = -\partial_2 P' + \nu\Delta w_2',\\&Sw_3' + u_2\partial_2 w_3' = -\partial_3 P' + \nu\Delta w_3'.\end{aligned}\right\} \tag{2.11}$$

Since the coefficients in (2.11) are periodic in space, it is natural to assume that the eigenfunctions correspond to an irreducible one-dimensional representation of the Abelian group of discrete translations $x_1 \to x_1 + nL_1$, $x_3 \to x_3 + mL_3$, n, m are integer. Therefore, we represent the solution in the form of Floquet–Bloch functions

$$\left.\begin{aligned}\boldsymbol{w}'(x_1,x_2,x_3) &= \boldsymbol{w}(x_1,x_3)\mathrm{e}^{\mathrm{i}(\boldsymbol{\kappa}\cdot\boldsymbol{r})},\\ P'(x_1,x_2,x_3) &= P(x_1,x_3)\mathrm{e}^{\mathrm{i}(\boldsymbol{\kappa}\cdot\boldsymbol{r})}\end{aligned}\right\} \tag{2.12}$$

where $\boldsymbol{\kappa} = (\kappa_1,\kappa_2,\kappa_3)$ is the wavevector and $\boldsymbol{r} = (x_1,x_2,x_3)$ is the radius vector. Because our goal is to find the sufficient conditions for the instability, we do not investigate here the completeness of the system of eigenfunctions, and do not consider functions different from (2.12). In (2.12), $\boldsymbol{w}(x_1,x_3)$ and $P(x_1,x_3)$ are spatially periodic functions

$$\begin{aligned}\boldsymbol{w}(x_1+L_1,x_3) &= \boldsymbol{w}(x_1,x_3+L_3) = \boldsymbol{w}(x_1,x_3),\\ P(x_1+L_1,x_3) &= P(x_1,x_3+L_3) = P(x_1,x_3).\end{aligned}$$

The wavevector $\boldsymbol{\kappa}$ is such that κ_2 is arbitrary, and κ_1 and κ_3 are defined modulo $2\pi/L_1$ and $2\pi/L_3$, respectively.

Substituting (2.12) into (2.11) we obtain

$$\left.\begin{aligned}&\nabla\cdot\boldsymbol{v} + \mathrm{i}\boldsymbol{\kappa}'\boldsymbol{v} + \mathrm{i}\kappa_2 v_2 = 0,\\&\boldsymbol{v}S + u_2\boldsymbol{v}\mathrm{i}\kappa_2 = -\nabla P + \nu\Delta\boldsymbol{v} + 2\mathrm{i}\nu(\boldsymbol{\kappa}'\cdot\nabla)\boldsymbol{v} - P\mathrm{i}\boldsymbol{\kappa}' - \nu\boldsymbol{v}(\boldsymbol{\kappa})^2,\\&v_2S + u_2v_2\mathrm{i}\kappa_2 + (\boldsymbol{v}\cdot\nabla)u_2 = \nu\Delta v_2 + 2\mathrm{i}\nu(\boldsymbol{\kappa}'\cdot\nabla)v_2 - P\mathrm{i}\kappa_2 - \nu v_2(\boldsymbol{\kappa})^2\end{aligned}\right\} \tag{2.13}$$

where $\boldsymbol{\kappa}' = (\kappa_1,\kappa_3)$, $\boldsymbol{v} = (v_1,v_3) = (w_1,w_3)$, $v_2 = w_2$.

2.3. *Long-wave asymptotics*

Since we are interested in long-wave asymptotics, take a small parameter $\epsilon = |\boldsymbol{\kappa}|$. We introduce the unit vector $\boldsymbol{n} = \boldsymbol{\kappa}/|\boldsymbol{\kappa}|$ ($|\boldsymbol{n}| = 1$), which determines the direction of the wavevector $\boldsymbol{\kappa}$. The series expansion of the growth rate S is

$$S = S^{(0)} + \epsilon S^{(1)} + \epsilon^2 S^{(2)} + \dots . \tag{2.14}$$

We are not interested in the whole spectrum of growth rates but only in the special mode with $S^{(0)} = 0$, which is responsible for the long-wave instability. We expand $\boldsymbol{v}$, v_2 and P in series

$$\left.\begin{aligned} \boldsymbol{v} &= \boldsymbol{v}^{(0)} + \epsilon \boldsymbol{v}^{(1)} + \epsilon^2 \boldsymbol{v}^{(2)} + \dots, \\ v_2 &= v_2^{(0)} + \epsilon v_2^{(1)} + \epsilon^2 v_2^{(2)} + \dots, \\ P &= P^{(0)} + \epsilon P^{(1)} + \epsilon^2 P^{(2)} + \dots \end{aligned}\right\} \tag{2.15}$$

with periodicity conditions

$$\left.\begin{aligned} \boldsymbol{v}^{(l)}(x_1 + L_1, x_3) &= \boldsymbol{v}^{(l)}(x_1, x_3 + L_3) = \boldsymbol{v}^{(l)}(x_1, x_3), \\ P^{(l)}(x_1 + L_1, x_3) &= P^{(l)}(x_1, x_3 + L_3) = P^{(l)}(x_1, x_3), \\ v_2^{(l)}(x_1 + L_1, x_3) &= v_2^{(l)}(x_1, x_3 + L_3) = v_2^{(l)}(x_1, x_3), \quad l = 0, 1, 2, \dots . \end{aligned}\right\} \tag{2.16}$$

Now we substitute expansions (2.14) and (2.15) into (2.13) and collect together the terms of the same order in ϵ. Thus, the leading-order problem is

$$(\nabla \cdot \boldsymbol{v}^{(0)}) = 0, \tag{2.17a}$$

$$\nabla P^{(0)} - \nu \Delta \boldsymbol{v}^{(0)} = 0, \tag{2.17b}$$

$$(\boldsymbol{v}^{(0)} \cdot \nabla) u_2 - \nu \Delta v_2^{(0)} = 0. \tag{2.17c}$$

Applying the divergence operator to (2.17*b*) we get that $\Delta P^{(0)} = 0$. Because of the spatial periodicity condition (2.16), we deduce that

$$P^{(0)} = \text{const}. \tag{2.18}$$

Now, (2.17*b*) and (2.18) give $\Delta \boldsymbol{v}^{(0)} = 0$, and one can obtain that $\boldsymbol{v}^{(0)} = \text{const}$ and

$$v_2^{(0)} = \nu^{-1} \Delta^{-1} (\boldsymbol{v}^{(0)} \cdot \nabla u_2) + \overline{v_2^{(0)}}, \tag{2.19}$$

where $\overline{v_2^{(0)}}$ is a constant, which will be determined from the system up to $O(\epsilon)$. Recall that according to our definition the mean value of the first term on the right-hand side of equation (2.19) is equal to zero.

To $O(\epsilon)$ we obtain

$$(\nabla \cdot \boldsymbol{v}^{(1)}) = -\mathrm{i}(\boldsymbol{v}^{(0)} \cdot \boldsymbol{q}) - \mathrm{i} v_2^{(0)} n_2, \tag{2.20a}$$

$$\nabla P^{(1)} - \nu \Delta \boldsymbol{v}^{(1)} = -S^{(1)} \boldsymbol{v}^{(0)} - \mathrm{i} u_2 \boldsymbol{v}^{(0)} n_2 - \mathrm{i} \boldsymbol{q} P^{(0)}, \tag{2.20b}$$

$$(\boldsymbol{v}^{(1)} \cdot \nabla) u_2 - \nu \Delta v_2^{(1)} = -S^{(1)} v_2^{(0)} - \mathrm{i} u_2 v_2^{(0)} n_2 - \mathrm{i} P^{(0)} n_2 + 2\mathrm{i}\nu (\boldsymbol{q} \cdot \nabla) v_2^{(0)}, \tag{2.20c}$$

where $\boldsymbol{q} = (n_1, n_3) = (\kappa_1/|\boldsymbol{\kappa}|, \kappa_3/|\boldsymbol{\kappa}|)$. The solvability condition for (2.20*a*) is $\langle -\mathrm{i}(\boldsymbol{v}^{(0)} \cdot \boldsymbol{q}) - \mathrm{i} v_2^{(0)} n_2 \rangle = 0$. Now the constant $\overline{v_2^{(0)}}$ is given by $\overline{v_2^{(0)}} = -(\boldsymbol{v}^{(0)} \cdot \boldsymbol{q})/n_2$ and (2.19) takes the form

$$v_2^{(0)} = \nu^{-1} \Delta^{-1} (\boldsymbol{v}^{(0)} \cdot \nabla u_2) - (\boldsymbol{v}^{(0)} \cdot \boldsymbol{q})/n_2.$$

Taking divergence of (2.20*b*) and performing simple calculations we get

$$P^{(1)} = -2\mathrm{i} n_2 \Delta^{-1} (\boldsymbol{v}^{(0)} \cdot \nabla u_2) + \mathrm{i}\nu (\boldsymbol{v}^{(0)} \cdot \boldsymbol{q}) + \overline{P^{(1)}}. \tag{2.21}$$

Note that Δ^{-1} in (2.21) is defined because of condition (2.6). Applying the operator of the gradient to (2.21) and using (2.20*b*) we obtain

$$\boldsymbol{v}^{(1)} = \nu^{-1} \Delta^{-1} \boldsymbol{T} + \overline{\boldsymbol{v}^{(1)}} \tag{2.22}$$

where

$$T = S^{(1)}\boldsymbol{v}^{(0)} - 2\mathrm{i}n_2\Delta^{-1}\left((\boldsymbol{v}^{(0)}\cdot\nabla)\nabla u_2\right) + u_2\mathrm{i}n_2\boldsymbol{v}^{(0)} + \mathrm{i}\boldsymbol{q}P^{(0)}.$$

The solvability condition for (2.22) is the existence condition for Δ^{-1}. Thus $\langle T\rangle$ has to be zero. The solvability condition is $S^{(1)}\boldsymbol{v}^{(0)} + \mathrm{i}\boldsymbol{q}P^{(0)} = 0$ and after multiplication by $\boldsymbol{v}^{(0)}$ and $\boldsymbol{q}$, we can write

$$\left.\begin{aligned} S^{(1)}(\boldsymbol{v}^{(0)}\cdot\boldsymbol{v}^{(0)}) + \mathrm{i}(\boldsymbol{q}\cdot\boldsymbol{v}^{(0)})P^{(0)} &= 0,\\ S^{(1)}(\boldsymbol{v}^{(0)}\cdot\boldsymbol{q}\,) + \mathrm{i}(\boldsymbol{q}\cdot\boldsymbol{q}\,)P^{(0)} &= 0.\end{aligned}\right\} \tag{2.23}$$

Finally,

$$\boldsymbol{v}^{(1)} = \mathrm{i}n_2\nu^{-1}\Delta^{-1}\left[u_2\boldsymbol{v}^{(0)} - 2\Delta^{-1}\left((\boldsymbol{v}^{(0)}\cdot\nabla)\nabla u_2\right)\right] + \overline{\boldsymbol{v}^{(1)}} \tag{2.24}$$

where $\overline{\boldsymbol{v}^{(1)}}$ will be calculated from the equations of $O(\epsilon^2)$. We rewrite (2.24) in the scalar form:

$$v_l^{(1)} = \mathrm{i}n_2\nu^{-1}\left[\delta_{lj}\Delta^{-1}u_2 - 2\nabla_j\Delta^{-2}(\nabla_l u_2)\right]v_j^{(0)} + \overline{v_l^{(1)}},\quad l = 1,3. \tag{2.25}$$

(Summing over the suppressed subscript j is used.) It follows from (2.20c) that

$$\begin{aligned} v_2^{(1)} = \nu^{-1}\Delta^{-1}\Big\{&\left[S^{(1)} + u_2\mathrm{i}n_2 - 2\mathrm{i}\nu(\boldsymbol{q}\cdot\nabla)\right]\nu^{-1}(\boldsymbol{v}^{(0)}\cdot\nabla u_2)\\ &- \left[S^{(1)} + u_2\mathrm{i}n_2 - 2\mathrm{i}\nu(\boldsymbol{q}\cdot\nabla)\right](\boldsymbol{v}^{(0)}\cdot\boldsymbol{q})/n_2 + P^{(0)}\mathrm{i}n_2 + (\boldsymbol{v}^{(1)}\cdot\nabla u_2)\Big\} + \overline{v_2^{(1)}},\end{aligned}$$

where $\overline{v_2^{(1)}}$ is a constant which will be determined from the system of $O(\epsilon^2)$. The solvability condition immediately implies

$$-S^{(1)}(\boldsymbol{v}^{(0)}\cdot\boldsymbol{q}\,)/n_2 + P^{(0)}\mathrm{i}n_2 = 0. \tag{2.26}$$

The conditions (2.23) and (2.26) form the system of homogeneous equations

$$\left.\begin{aligned} S^{(1)}(\boldsymbol{v}^{(0)}\cdot\boldsymbol{v}^{(0)}) + \mathrm{i}(\boldsymbol{q}\cdot\boldsymbol{v}^{(0)})P^{(0)} &= 0,\\ S^{(1)}(\boldsymbol{v}^{(0)}\cdot\boldsymbol{q}\,) + \mathrm{i}(\boldsymbol{q}\cdot\boldsymbol{q}\,)P^{(0)} &= 0,\\ S^{(1)}(\boldsymbol{v}^{(0)}\cdot\boldsymbol{q}\,) - \mathrm{i}n_2^2P^{(0)} &= 0.\end{aligned}\right\} \tag{2.27}$$

The system (2.27) is overdetermined and it has only the zero solution $S^{(1)} = P^{(0)} = 0$. Thus,

$$\begin{aligned} v_2^{(1)} = \nu^{-1}\Delta^{-1}\Big\{&[u_2\mathrm{i}n_2 - 2\mathrm{i}\nu(\boldsymbol{q}\cdot\nabla)]\,\nu^{-1}(\boldsymbol{v}^{(0)}\cdot\nabla u_2)\\ &- [u_2\mathrm{i}n_2 - 2\mathrm{i}\nu(\boldsymbol{q}\cdot\nabla)](\boldsymbol{v}^{(0)}\cdot\boldsymbol{q})/n_2 + (\boldsymbol{v}^{(1)}\cdot\nabla u_2)\Big\} + \overline{v_2^{(1)}}.\end{aligned}$$

After some calculations, we get

$$\begin{aligned} v_2^{(1)} = \mathrm{i}\{&\nu^{-2}n_2\nabla_j(u_2\Delta^{-1}u_2) \;-\; 2\nu^{-1}n_m\Delta^{-2}\nabla_m\nabla_j u_2\\ &-\nu^{-1}n_j\Delta^{-1}u_2 - 2\nu^{-1}n_2\Delta^{-1}(\Delta^{-2}(\nabla_j\nabla_m u_2)\nabla_m u_2)\}v_j^{(0)} + \nu^{-1}\Delta^{-1}\nabla u_2\overline{\boldsymbol{v}^{(1)}} + \overline{v_2^{(1)}}.\end{aligned} \tag{2.28}$$

(Summing over the suppressed subscripts m and j is used.)

Up to $O(\epsilon^2)$ we obtain

$$(\nabla\cdot\boldsymbol{v}^{(2)}) = -\mathrm{i}(\boldsymbol{v}^{(1)}\cdot\boldsymbol{q}\,) - \mathrm{i}v_2^{(1)}n_2, \tag{2.29a}$$

$$\nabla P^{(2)} - \nu\Delta\boldsymbol{v}^{(2)} = -S^{(1)}\boldsymbol{v}^{(0)} - \mathrm{i}u_2\boldsymbol{v}^{(1)}n_2 - \mathrm{i}\boldsymbol{q}P^{(1)} - \nu\boldsymbol{v}^{(0)}, \tag{2.29b}$$

$$(\boldsymbol{v}^{(2)}\cdot\nabla)u_2 - \nu\Delta v_2^{(2)} = -S^{(2)}v_2^{(0)} - \mathrm{i}u_2v_2^{(1)}n_2 - \mathrm{i}P^{(1)}n_2 + 2\mathrm{i}\nu(\boldsymbol{q}\cdot\nabla)v_2^{(1)} - \nu v_2^{(0)}. \tag{2.29c}$$

The solvability condition for (2.29a) is

$$\langle(\boldsymbol{v}^{(1)}\cdot\boldsymbol{q}\,) + v_2^{(1)}n_2\rangle = 0.$$

Thus $\overline{v_2^{(1)}} = -(\boldsymbol{v}^{(1)} \cdot \boldsymbol{q})/n_2$. Taking divergence of (2.29*b*) we obtain

$$\begin{aligned} P^{(2)} = \;& (\nu\nabla \cdot \boldsymbol{v}^{(2)}) \\ &+ \mathrm{i}\Delta^{-1}\left\{-(\nabla \cdot (\boldsymbol{q}P^{(1)})) - n_2(\nabla \cdot (u_2\boldsymbol{v}^{(1)})) + 2\nu(\nabla \cdot ((\boldsymbol{q}\cdot\nabla)\boldsymbol{v}^{(1)}))\right\} + \overline{P^{(2)}}, \end{aligned}$$

and an additional solvability condition does not appear because the expressions with Δ^{-1} contain divergence only but not a constant. We express $\boldsymbol{v}^{(2)}$ from (2.29*b*):

$$\boldsymbol{v}^{(2)} = \nu^{-1}\Delta^{-1}(S^{(2)}\boldsymbol{v}^{(0)} + u_2\boldsymbol{v}^{(1)}\mathrm{i}n_2 + \nabla P^{(2)} + \mathrm{i}\boldsymbol{q}P^{(1)} - 2\mathrm{i}\nu(\boldsymbol{q}\cdot\nabla)\boldsymbol{v}^{(1)} + \nu\boldsymbol{v}^{(0)}) + \overline{\boldsymbol{v}^{(2)}}.$$

The solvability condition is

$$S^{(2)}\boldsymbol{v}^{(0)} + \mathrm{i}n_2\langle u_2\boldsymbol{v}^{(1)}\rangle + \mathrm{i}\boldsymbol{q}\overline{P^{(1)}} + \nu\boldsymbol{v}^{(0)} = 0. \tag{2.30}$$

Now we express $v_2^{(2)}$ from (2.29*c*):

$$v_2^{(2)} = \nu^{-1}\Delta^{-1}\{S^{(2)}v_2^{(0)} + (\boldsymbol{v}^{(2)}\cdot\nabla u_2) + u_2v_2^{(1)}\mathrm{i}n_2 + P^{(1)}\mathrm{i}n_2 - 2\mathrm{i}\nu(\boldsymbol{q}\cdot\nabla)v_2^{(1)} + \nu v_2^{(0)}\}.$$

The solvability condition is

$$S^{(2)}\overline{v_2^{(0)}} + \mathrm{i}n_2\langle u_2v_2^{(1)}\rangle + \mathrm{i}n_2\overline{P^{(1)}} + \nu\overline{v_2^{(0)}} + \langle(\boldsymbol{v}^{(2)}\cdot\nabla u_2)\rangle = 0. \tag{2.31}$$

Using the notation

$$\tau = S^{(2)} + \nu, \tag{2.32}$$

we obtain from (2.30), (2.31) that

$$\left.\begin{aligned} &\tau\boldsymbol{v}^{(0)} + \mathrm{i}n_2\langle u_2\boldsymbol{v}^{(1)}\rangle + \mathrm{i}\boldsymbol{q}\overline{P^{(1)}} = 0, \\ &\mathrm{i}\tau(\boldsymbol{v}^{(0)}\cdot\boldsymbol{q}) + n_2(\boldsymbol{q}\cdot\langle u_2\boldsymbol{v}^{(1)}\rangle) + 2n_2^2\langle u_2v_2^{(1)}\rangle + \mathrm{i}n_2^2\overline{P^{(1)}} = 0. \end{aligned}\right\} \tag{2.33}$$

After simple calculations one can obtain from (2.25) and (2.28) that

$$\begin{aligned} &\langle u_2v_l^{(1)}\rangle = \mathrm{i}\nu^{-1}n_2A_{lj}v_j^{(0)}, \\ &\langle u_2v_2^{(1)}\rangle = \mathrm{i}(\nu^{-2}n_2M_j + \nu^{-1}q_mN_j)v_j^{(0)}, \quad j = 1, 3\ , \end{aligned}$$

where

$$A_{lj} = \langle u_2(\delta_{lj}\Delta - 2\nabla_j\nabla_l)\Delta^{-2}u_2\rangle, \tag{2.34}$$

$$M_j = \langle u_2\Delta^{-1}\left\{\nabla_j(u_2\Delta^{-1}u_2) - 2\left(\Delta^{-2}(\nabla_j\nabla_mu_2)\cdot\nabla_mu_2\right)\right\}\rangle, \tag{2.35}$$

$$N_{mj} = -\langle u_2(2\nabla_m\nabla_j + \delta_{mj}\Delta)\Delta^{-2}u_2\rangle, \quad j = 1, 3. \tag{2.36}$$

The system (2.33) takes the form

$$\left.\begin{aligned} &(\tau\delta_{lj} - \nu^{-1}n_2^2A_{lj})v_j^{(0)} + q_l\overline{P^{(1)}} = 0, \\ &(\tau q_j - \nu^{-1}\mathrm{i}n_2^2q_mA_{mj} + 2\nu^{-1}\mathrm{i}n_2^3M_j + 2\nu^{-1}n_2^2q_mN_{mj})v_j^{(0)} - \mathrm{i}n_2\overline{P^{(1)}} = 0, \end{aligned}\right\} \tag{2.37}$$

$l, j, m = 1, 3$ and summing over the suppressed subscripts m is supposed.

Equation (2.37) is a system of homogeneous equations in $v_j^{(0)}, \overline{P^{(1)}}$. After elimination of $\overline{P^{(1)}}$ it can be written in the form

$$\left\{\tau(\delta_{lj} + n_2^{-2}n_ln_j) + \nu^{-1}(-A_{lj}n_2^2 + n_ln_mG_{mj} + 2n_2n_lM_j\nu^{-1})\right\}v_j^{(0)} = 0, \tag{2.38}$$

where

$$G_{mj} = A_{mj} + 2N_{mj}. \tag{2.39}$$

The system (2.38) has non-zero solutions if the determinant of (2.38) is equal to zero:

$$\det\left\{\tau(\delta_{lj}+n_2^{-2}n_ln_j)+\nu^{-1}(-A_{lj}n_2^2+n_ln_mG_{mj}+2n_2n_lM_j\nu^{-1})\right\}=0, \tag{2.40}$$

$l,j,m=1,3$ and summing over the suppressed subscripts m is supposed.

The formula (2.40), which determines the long-wave asymptotics of growth rates, is the main result of this paper. The rest of the paper is devoted to applications of this formula.

In almost all the previous investigations the wavevector $\boldsymbol{\kappa}$ was taken to be parallel to the direction of the flow, i.e. $\boldsymbol{n}=(0,1,0)$. In this particular case the stability criterion (2.40) takes the simple form $\det[\tau\delta_{lj}-\nu^{-1}A_{lj}]=0, \quad l,j=1,3$, or

$$\tau^2\nu^2-\tau\nu\ \mathrm{tr}\boldsymbol{A}+\det\boldsymbol{A}=0. \tag{2.41}$$

3. Flows periodic in one direction

3.1. *The Kolmogorov flow*

The first particular case is the Kolmogorov flow (Meshalkin & Sinai 1961) given by

$$u_1=0,$$
$$u_2=\sin x_1.$$

For this flow the matrix A_{lj} is defined in (2.34) takes the form: $A_{11}=-A_{33}=1/2, A_{13}=A_{31}=0$. The criterion (2.41) now becomes:

$$\tau^2\nu^2-1/4=0. \tag{3.1}$$

The roots of (3.1) are $\tau_{1,2}=\pm\frac{1}{2}\nu^{-1}$. Using (2.32) we obtain

$$S_1^{(2)}=-\frac{1}{2\nu}-\nu<0, \tag{3.2a}$$

$$S_2^{(2)}=\frac{1}{2\nu}-\nu. \tag{3.2b}$$

The mode corresponding to $S_1^{(2)}$ as in (3.2*a*) is stable, and the mode corresponding to $S_2^{(2)}$ as in (3.2*b*) can give instability. The critical value of ν (that is, $S_2^{(2)}=0$) is $\nu_{cr}=2^{-1/2}$. This is identical with the result of Meshalkin & Sinai (1961).

We now investigate the Kolmogorov flow in the general case $\boldsymbol{n}=(n_1,n_2,n_3)$. The matrices A_{lj} (2.34), G_{lj} (2.39), and vector M_j (2.35) take the form

$$\boldsymbol{A}=\begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix},\ \boldsymbol{G}=\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{7}{2}\end{pmatrix},\ \boldsymbol{M}=\begin{pmatrix}0\\0\end{pmatrix}.$$

The criterion (2.40) now is

$$\tau^2n_2^{-2}+4\tau n_1^2\nu^{-1}-n_2^2(1-8n_1^2)(2\nu)^{-2}=0. \tag{3.3}$$

The roots of (3.3) are $\tau_{1,2}=-n_2^2\nu^{-1}(2n_1^2\pm 2n_1^2\mp\frac{1}{2})$. Using (2.32) we obtain

$$S_1^{(2)}=-n_2^2(2\nu)^{-1}-\nu<0, \tag{3.4a}$$

$$S_2^{(2)}=-n_2^2\nu^{-1}(4n_1^2-1/2)-\nu. \tag{3.4b}$$

The mode corresponding to $S_1^{(2)}$ is stable and the mode corresponding to $S_2^{(2)}$ (3.4*b*) can give instability. Noting that the modes depend on n_1, n_2 only we introduce the angle φ between the x_2-axis (the direction of the flow) and the direction of wavevector $\boldsymbol{\kappa}$, characterizing the of the wavevector: $n_1 = \sin\varphi, n_2 = \cos\varphi$ and seek angle the positive maximum of the function $S_2^{(2)}(\varphi)$ where $\varphi \in [0, 2\pi]$. A simple calculation shows that

$$\max_{\varphi\in[0,2\pi]} S_2^{(2)}(\varphi) = S_2^{(2)}(0) \equiv S_2^{(2)}(\pi) = (2\nu)^{-1} - \nu.$$

Thus, the critical value of ν is $2^{-1/2}$, which is attained at $n_1 = 0, n_2 = \pm 1$.

So the direction parallel to the flow is the most dangerous one. Therefore, the result of Meshalkin & Sinai (1961) is sufficient.

3.2. *The result of Yudovich*

The flow

$$\left.\begin{aligned} u_1 &= 0, \\ u_2 &= -\partial_1 \psi(x_1) \end{aligned}\right\} \tag{3.5}$$

where $\psi(x_1)$ is an arbitrary function, was first investigated by Yudovich (1966) in the particular case where the wavevector $\boldsymbol{\kappa}$ is parallel to the direction of the flow ($\boldsymbol{n} = (0, 1, 0)$). The matrix A_{ij} (2.34) is $A_{11} = -A_{33} = \lambda, A_{13} = A_{31} = 0$, where

$$\lambda = \langle \psi^2 \rangle = \frac{1}{L_1} \int_0^{L_1} \psi^2(x_1)\mathrm{d}x_1. \tag{3.6}$$

The criterion (2.41) now takes the form

$$\tau^2 \nu^2 - \lambda^2 = 0. \tag{3.7}$$

The roots of (3.7) are $\tau_{1,2} = \pm\lambda\nu^{-1}$. Using (2.32) we obtain

$$S_1^{(2)} = -\lambda\nu^{-1} - \nu < 0, \tag{3.8a}$$

$$S_2^{(2)} = \lambda\nu^{-1} - \nu. \tag{3.8b}$$

The mode corresponding to $S_1^{(2)}$ (3.8*a*) is a stable one and the mode corresponding to $S_2^{(2)}$ (3.8*b*) can give non-stability. The critical value of ν is $\lambda^{1/2}$. It is identical with the result of Yudovich (1966) for the flow (3.5).

3.3. *The analysis of Dubrulle & Frisch*

For the considered flow (3.5) the matrices A_{ij} (2.34), G_{ij} (2.39) and vector M_j (2.35) take the form:

$$\boldsymbol{A} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \quad \boldsymbol{G} = \begin{pmatrix} 7\lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad \boldsymbol{M} = \begin{pmatrix} 3\mu \\ 0 \end{pmatrix},$$

where λ is defined by (3.6) and

$$\mu = \tfrac{1}{2}\langle \psi^3 \rangle = \frac{1}{2L_1} \int_0^{L_1} \psi^3(x_1)\mathrm{d}x_1. \tag{3.9}$$

The criterion (2.40) now takes the form

$$\tau^2 n_2^{-2} + 2\tau\nu^{-1}T - n_2^2\nu^{-2}\lambda(\lambda - 2T) = 0 \tag{3.10}$$

where $T = 4\lambda n_1^2 + 3\mu\nu^{-1}n_1 n_2$. The roots of (3.10) are

$$\tau_{1,2} = n_2^2\nu^{-1}(-T \pm T \mp \lambda)$$

or, in terms of modes,

$$S_1^{(2)} = -n_2^2 v^{-1}\lambda - v < 0 \qquad (3.11a)$$

$$S_2^{(2)} = n_2^2 v^{-1}(\lambda - 2T) - v. \qquad (3.11b)$$

The mode corresponding to $S_1^{(2)}$ (3.11*a*) is the stable one and the mode corresponding to $S_2^{(2)}$ (3.11*b*) can give instability.

This is identical with the result of Dubrulle & Frisch (1991) for this flow.

Under the supposition that $\boldsymbol{n} = (0, 1, 0)$ (3.11*b*) takes the form $S_2^{(2)} = \lambda v^{-1} - v$, which is identical to Yudovich (1966), but this direction is not always the dangerous one, as will be shown on the simple example in the next subsection.

3.4. *The flow* $u_2 = \sin x_1 + \sigma \sin 2x_1$

The particular case of the following one-dimensional basic flow was investigated by Henon & Scholl (1991): $u_1 = 0$, $u_2 = \sin x_1 + \sigma \sin 2x_1$. For this flow the parameters λ (see (3.6)) and μ (see (3.9)) are: $\lambda = (4+\sigma^2)/8, \mu = 3\sigma/16$. The expression for $S_2^{(2)}$ corresponding to a non-stable mode (3.11*b*) is

$$S_2^{(2)} = n_2^2\left[(4+\sigma^2)(1-8n_1^2) - 9\sigma n_1 n_2/v\right]/(8v) - v.$$

To show the existence of an instability we introduce the spherical coordinates

$$\left.\begin{aligned} n_1(\theta,\varphi) &= \sin\theta\cos\varphi, \\ n_2(\theta,\varphi) &= \cos\theta, \\ n_3(\theta,\varphi) &= \sin\theta\sin\varphi. \end{aligned}\right\} \qquad (3.12)$$

Thus,

$$S_2^{(2)}(\theta,\varphi) = \cos^2\theta\left[(4+\sigma^2)(1-8\sin^2\theta\cos^2\varphi) - 9\sigma\sin\theta\cos\varphi\cos\theta/v\right]/(8v).$$

Using the software *Mathematica* one can obtain that for $\sigma = \sqrt{2}$ the maximum of $S_2^{(2)}(\theta,\varphi)$ is equal to zero, and attained at $\theta = 7.014°, \varphi = 0°$, i.e. the turn of wavevector is 7°, where the critical value of v is equal to 0.925197. This is identical with the result from Henon & Scholl (1991). The plot of $S_2^{(2)}(\theta,\varphi)$ for the critical value of v is shown on figure 1.

3.5. *Wavevector of the most dangerous disturbances*

As it was shown above, the mode (3.11*b*) corresponding to

$$S_2^{(2)} = n_2^2(\lambda - 8n_1^2\lambda - 6\mu v^{-1}n_1n_2)/v - v$$

can give instability. Noting that the growth rate depends on n_1, n_2 only, we introduce the angle φ characterizing the direction of the wavevector: $n_1 = \sin\varphi$, $n_2 = \cos\varphi$ where $\varphi \in [0, 2\pi]$. Then

$$S_2^{(2)}(\varphi) = v^{-1}\cos^2\varphi(\lambda - 8\lambda\sin^2\varphi - 6\mu v^{-1}\sin\varphi\cos\varphi) - v,$$

or, after simplification,

$$S_2^{(2)}(\varphi) = \lambda(2\cos 4\varphi + \cos 2\varphi - 1)/(2v) - 3\mu(\sin 4\varphi + 2\sin 2\varphi)/(4v^2) - v. \qquad (3.13)$$

The conditions for existence of the maximum are

$$\left.\begin{aligned} \mathrm{d}S_2^{(2)}/\mathrm{d}\varphi &= -\lambda(4\sin 4\varphi + \sin 2\varphi)/v + 3\mu(\cos 4\varphi + \cos 2\varphi)/v^2 = 0, \\ \mathrm{d}^2S_2^{(2)}/\mathrm{d}\varphi^2 &= -2\lambda(8\cos 4\varphi + \cos 2\varphi)/v - 6\mu(2\sin 4\varphi + \sin 2\varphi)/v^2 < 0. \end{aligned}\right\} \qquad (3.14)$$

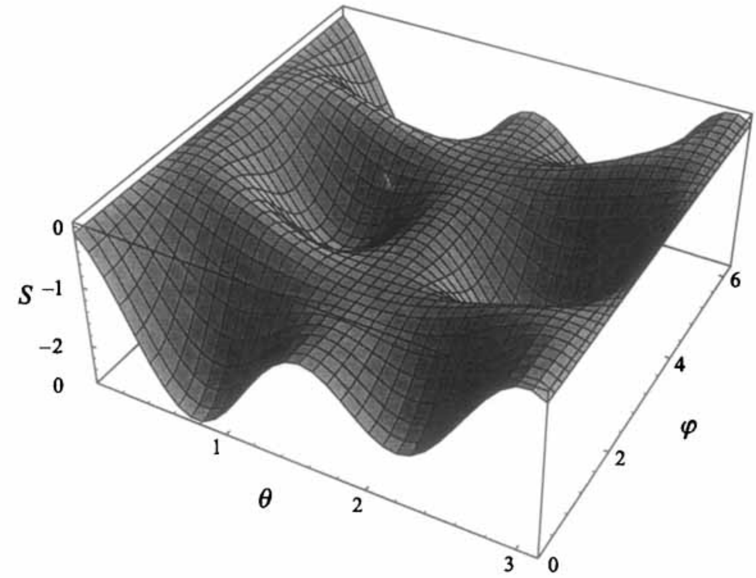

FIGURE 1. The three-dimensional plot of the growth rate $S_2^{(2)}(\theta, \varphi)$ corresponding to the unstable mode for the flow $u_2 = \sin x_1 + \sqrt{2} \sin 2x_1$.

Equation (3.14*a*) has the trivial solution $\varphi = \pi/2$, which does not give a positive maximum because $S_2^{(2)}(\pi/2) = -\nu < 0$. The system of equations (3.14) may be transformed to the form

$$4\sin 3\varphi + \eta \cos 3\varphi - 3\sin\varphi = 0, \qquad (3.15a)$$

$$8\cos 4\varphi + \cos 2\varphi + \eta(2\sin 4\varphi + \sin 2\varphi) > 0, \qquad (3.15b)$$

where $\eta = 3\mu/(\lambda\nu)$. Equation (3.15*a*) is simplified by the substitution $y = e^{2i\varphi}$, which transforms it into

$$(\eta \mathrm{i} + 4)y^3 - 3y^2 + 3y + (\eta \mathrm{i} - 4) = 0.$$

The result of Yudovich (1966) for this flow is obtained if the system (3.15) has no solutions in the region, i.e. the maximum is exceeded on the boundary of the region $\varphi = 0$, and from (3.13) we obtain $S_2^{(2)}(0) = \lambda/\nu - \nu$ and the critical value of ν is $\lambda^{1/2}$.

4. Flows periodic in two directions

4.1. *Shtilman & Sivashinsky flow*

The flow

$$u_1 = 0,$$
$$u_2 = \sin\alpha x_1 \, \sin\beta x_3, \quad \alpha^2 + \beta^2 = 1$$

was investigated by Shtilman & Sivashinsky (1986) in case where the wavevector $\boldsymbol{\kappa}$ is parallel to the direction of the flow, i.e. $\boldsymbol{n} = (0, 1, 0)$. In this case the matrix A_{ij} (2.34) takes the form

$$\boldsymbol{A} = \begin{pmatrix} \frac{1}{4}(\alpha^2 - \beta^2) & 0 \\ 0 & -\frac{1}{4}(\alpha^2 - \beta^2) \end{pmatrix}.$$

Thus, criterion (2.41) gives two growth rates:

$$S_1^{(2)} = -\frac{1}{4\nu}(\alpha^2 - \beta^2) - \nu, \tag{4.1a}$$

$$S_2^{(2)} = \frac{1}{4\nu}(\alpha^2 - \beta^2) - \nu. \tag{4.1b}$$

If $\alpha^2 > \beta^2$, the mode corresponding to (4.1*a*) is stable, and the mode corresponding to (4.1*b*) is unstable, and if $\alpha^2 < \beta^2$ the mode corresponding to (4.1*a*) is unstable and the second one (4.1*b*) is stable. It is identical with the result of Shtilman & Sivashinsky (1986).

The Shtilman–Sivashinsky flow is investigated in the case where $\boldsymbol{n} = (n_1, n_2, n_3)$. The spherical coordinates are introduced by the formulae (3.12). The necessary conditions for the existence of a positive maximum of S are

$$\tau^2 + a(\theta, \varphi)\tau + b(\theta, \varphi) = 0, \tag{4.2a}$$

$$a_\theta \tau + b_\theta = 0, \tag{4.2b}$$

$$a_\varphi \tau + b_\varphi = 0 \tag{4.2c}$$

where $a(\theta, \varphi) = \frac{1}{2}\nu^{-1} \sin^2 2\theta(\alpha^2 \cos^2 \varphi + \beta^2 \sin^2 \varphi)$,
$b(\theta, \varphi) = \frac{1}{16}\nu^{-2} \cos^4 \theta[-(\alpha^2 - \beta^2)^2 + 8(\alpha^2 - \beta^2) \sin^2 \theta(\alpha^2 \cos^2 \varphi - \beta^2 \sin^2 \varphi)]$,
$a_\theta = \partial a/\partial\theta, \quad b_\theta = \partial b/\partial\theta, \quad a_\varphi = \partial a/\partial\varphi, \quad b_\varphi = \partial b/\partial\varphi.$

Let us first consider (4.2*c*) in the case where $a_\varphi \neq 0$. Then

$$\tau = -b_\varphi/a_\varphi = -\cos^2\theta/(4\nu) < 0,$$

and it cannot give a positive maximum. If

$$a_\varphi = \frac{1}{2\nu} \sin^2 2\theta \sin 2\varphi(\beta^2 - \alpha^2) = 0, \tag{4.3}$$

then it is easy to see from (4.2*c*) that

$$b_\varphi = \frac{1}{2\nu^2} \cos^4\theta \sin^2\theta \sin 2\varphi(\beta^2 - \alpha^2) = 0. \tag{4.4}$$

Equations (4.3) and (4.4) are equivalent to $\varphi = 0, \pi/2, \pi, 3\pi/2, 2\pi$ or $\theta = 0, \pi/2, \pi$. The points with $\theta = 0, \pi$ give $a = 0, b = -\frac{1}{16}\nu^{-2}(\alpha^2 - \beta^2)^2$ and

$$\tau_+ = \frac{1}{4\nu}|\alpha^2 - \beta^2|.$$

The points with $\theta = \pi/2$ give $a = 0, b = 0$ and $\tau = 0$. In what follows it may be supposed without loss of generality that $\alpha < \beta$. Thus, we obtain for points with $\varphi = 0, \pi, 2\pi$

$$a = \alpha^2 \sin^2 2\theta/(2\nu),$$
$$b = \cos^4\theta[-(\alpha^2 - \beta^2)^2 + 8\alpha^2(\alpha^2 - \beta^2)\sin^2\theta]/(16\nu^2).$$

The positive root is

$$\tau_+ = \cos^2\theta(\beta^2 - \alpha^2)/(4\nu)$$

and its maximum is

$$\max_{\theta=0} \tau_+ = (\beta^2 - \alpha^2)/(4\nu).$$

For points with $\varphi = \pi/2, 3\pi/2$ we obtain

$$a = \beta^2 \sin^2 2\theta/(2v),$$
$$b = \cos^4\theta[-(\alpha^2-\beta^2)^2 + 8\beta^2(\alpha^2-\beta^2)\sin^2\theta]/(16v^2)$$

and the positive root is

$$\tau_+ = \cos^2\theta(-\beta^2\sin^2\theta + |\beta^2\sin^2\theta + (\alpha^2-\beta^2)/4|)/v.$$

There are two possibilities. The first is that $\sin^2\theta \geqslant (\beta^2-\alpha^2)/(4\beta^2)$ with

$$\tau_+ = \cos^2\theta(\alpha^2-\beta^2)/(4v) < 0.$$

The second one is that $\sin^2\theta \leqslant (\beta^2-\alpha^2)(4\beta^2)$ with

$$\tau_+ = \cos^2\theta[(\beta^2-\alpha^2)/4 - 2\sin^2\theta\beta^2)/v$$

and the obvious maximum

$$\max_{\theta=0}\tau_+ = (\beta^2-\alpha^2)/(4v).$$

Thus, the Shtilman–Sivashinsky result is also true in the three-dimensional case.

4.2. *Results of Brutyan & Krapivsky*

Brutyan & Krapivsky investigated the stability of the flow $u_1 = 0, u_2 = u_2(x_1, x_3)$ in the particular case where the wavevector is parallel to the direction of the flow $\boldsymbol{n} = (0,1,0)$. For this flow the matrix A_{lj} (2.34) takes the form

$$A_{lj} = \delta_{lj}B - 2B_{lj}$$

where $B = \langle u_2\Delta^{-1}u_2\rangle$, $B_{lj} = \langle u_2\partial^2_{lj}(\Delta^{-2}u_2)\rangle$. The eigenvalues are

$$\tau_{1,2} = \pm v^{-1}(B^2 - 4B_{11}B_{33} + 4B_{13}^2)^{1/2}.$$

The growth rate for the unstable mode is

$$S_2^{(2)} = v^{-1}(B^2 - 4B_{11}B_{33} + 4B_{13}^2)^{1/2} - v$$

and the critical parameter:

$$v_{cr} = (B^2 - 4B_{11}B_{33} + 4B_{13}^2)^{1/4}, \tag{4.5}$$

which is identical to the criterion of Brutyan & Krapivsky (1991).

Taking into account the abovementioned results of Dubrulle & Frisch (see §3.3), we can conclude that the investigation of Brutyan & Krapivsky giving (4.5) is incomplete because considering the case $\boldsymbol{\kappa} = (0, \kappa, 0)$ is not enough for a full stability analysis.

4.3. *Oscillatory instability*

Let us consider the following particular case:

$$\left.\begin{aligned} u_1 &= 0,\\ u_2 &= -\partial_1\psi_1(x_1) - \partial_3\psi_3(x_3),\\ u_3 &= 0. \end{aligned}\right\} \tag{4.6}$$

In this case the matrices A_{lj} (2.34), G_{lj} (2.39) and vector M_j (2.35) take the form:

$$\boldsymbol{A} = \begin{pmatrix} \lambda_1-\lambda_3 & 0 \\ 0 & \lambda_3-\lambda_1 \end{pmatrix}, \quad \boldsymbol{G} = \begin{pmatrix} 7\lambda_1+\lambda_3 & 0 \\ 0 & \lambda_1+\lambda_3 \end{pmatrix}, \quad \boldsymbol{M} = \begin{pmatrix} 3\mu_1 \\ 3\mu_3 \end{pmatrix}$$

where

$$\lambda_j = \langle \psi_j^2 \rangle = \frac{1}{L_j} \int_0^{L_j} \psi_j^2(x_j)\mathrm{d}x_j, \tag{4.7a}$$

$$\mu_j = \tfrac{1}{2}\langle \psi_j^3 \rangle = \frac{1}{2L_j} \int_0^{L_j} \psi_j^3(x_j)\mathrm{d}x_j, \quad j = 1, 3. \tag{4.7b}$$

The criterion (2.40) takes the form

$$\tau^2 n_2^{-2} + 2\tau v^{-1}(T_1 + T_3) + n_2^2 v^{-2}[-(\lambda_1 - \lambda_3)^2 + 2(\lambda_1 - \lambda_3)(T_1 - T_3)] = 0 \tag{4.8}$$

where

$$T_i = 4\lambda_i n_i^2 + 3\mu_i v^{-1} n_2 n_i, \quad i = 1, 3. \tag{4.9}$$

(Summing over suppressed subscripts is not supposed.)

The flow (4.6) is the simplest generalization of the one-dimensional flow (3.9) considered in §3.3. However, in this case one can find some new features of the behaviour of long-wave disturbances. The most interesting difference between flows (4.6) and (3.9) is the following one. In contrast to equation (3.11), the roots of equation (4.8) can be *complex*.

Moreover, if the real part of τ is larger than v, we obtain a new type of oscillatory long-wavelength instability, which is characterized by complex eddy viscosity with negative real part.

We want to prove this for a particular case of the flow (4.6). The conditions for such instability are

$$T_1 + T_3 < -v^2 n_2^{-2}, \tag{4.10a}$$

$$(T_1 + T_3)^2 + (\lambda_1 - \lambda_3)^2 - 2(\lambda_1 - \lambda_3)(T_1 - T_3) < 0, \tag{4.10b}$$

which represent the conditions for existence of complex roots with positive real part for equation (4.8).

4.3.1. *A special case of oscillatory instability*

Let us consider the special subclasses of (4.6) with

$$\left.\begin{aligned} &\lambda_3 > 3/2, \quad \lambda_3/3 < \lambda_1 < \lambda_3 - 1, \\ &0 < \mu_3^2 < (3\lambda_1 - \lambda_3)\lambda_3^2/[2\lambda_1 + \lambda_3(\lambda_3 - \lambda_1)], \\ &\mu_1^2 = 2\lambda_1(1 + \mu_3^2/\lambda_3), \end{aligned}\right\} \tag{4.11}$$

and the following direction:

$$n_1 = -n_2\mu_1/(2v\lambda_1), \tag{4.12a}$$

$$n_2 = v(\lambda_3 - \lambda_1)^{1/2}, \tag{4.12b}$$

$$n_3 = -n_2\mu_3/(v\lambda_3). \tag{4.12c}$$

From condition $|\boldsymbol{n}| = 1$ we obtain that v must be

$$v = [1/(\lambda_3 - \lambda_1) - \mu_1^2/(2\lambda_1)^2 - \mu_3^2/\lambda_3^2]^{1/2}. \tag{4.13}$$

(For the selected parameters $\lambda_1, \lambda_3, \mu_1, \mu_3$ the expression under the square root is positive.) For direction (4.12) T_1 and T_3 take the following simple form

$$\left.\begin{aligned} T_1 &= -n_2^2\mu_1^2/(2\lambda_1 v^2), \\ T_3 &= \ \ n_2^2\mu_3^2/(\lambda_3 v^2). \end{aligned}\right\} \tag{4.14}$$

Thus, $T_1 + T_3 = -n_2^2 v^{-2}[\mu_1^2(2\lambda_1)^{-1} - \mu_3^2\lambda_3^{-1}]$, or because of (4.12*c*):

$$T_1 + T_3 = -n_2^2 v^{-2} < -v^2 n_2^{-2}.$$

The last inequality is true because

$$n_2^2/v^2 = \lambda_3 - \lambda_1 > 1. \tag{4.15}$$

For the justification of (4.10*b*), we rewrite it in the following form:

$$[T_1 + T_3 + (\lambda_3 - \lambda_1)]^2 - 4(\lambda_3 - \lambda_1)T_3 < 0.$$

Using (4.14) and (4.15), we obtain

$$-4(\lambda_3 - \lambda_1)^2 \mu_3^2/\lambda_3 < 0,$$

which is identically true. Thus, for the wide class of flows in the specific direction (4.12) the oscillatory instability exists.

4.3.2. *The flow* $u_2 = \sigma_1 \sin x_1 + \sigma_2 \sin 2x_1 + \sigma_3 \sin x_3 + \sigma_4 \sin 2x_3$

To illustrate the previous result we investigate the flow

$$\left.\begin{aligned} &u_1 = 0, \\ &u_2 = \sigma_1 \sin x_1 + \sigma_2 \sin 2x_1 + \sigma_3 \sin x_3 + \sigma_4 \sin 2x_3, \\ &u_3 = 0. \end{aligned}\right\} \tag{4.16}$$

For this flow the parameters λ_i and $\mu_i, i = 1, 3$, (4.7) take the following simple form: $\lambda_i = (4\sigma_i^2 + \sigma_{i+1}^2)/8$, $\mu_i = 3\sigma_i^2\sigma_{i+1}/16, i = 1, 3$. If we select σ_j, $j = 1, ..., 4$ such that conditions (4.11) for λ_i, $\mu_i, i = 1, 3$, hold then we can obtain oscillatory instability in the fixed direction (4.12).

Let λ_i and $\mu_i, i = 1, 3$, be $\lambda_1 = 18, \mu_1 = 6\sqrt{2}, \lambda_3 = 20, \mu_3 = 2\sqrt{5}$, which corresponds to

$$\sigma_1 \approx 2, \ \sigma_2 \approx 11.3, \ \sigma_3 \approx 6.3, \ \sigma_4 \approx 0.6. \tag{4.17}$$

From (4.13) we obtain the parameter v

$$v = 0.6. \tag{4.18}$$

Flow (4.16) with (4.17) and v given by (4.18) has a complex growth rate with positive real part for the direction $n_1 = -0.3, n_2 = 0.9, n_3 = -0.3$. From (4.8) and (2.32) we obtain the growth rate

$$S^{(2)} = \left\{\pm \left[(T_1 + T_3)^2 + (\lambda_1 - \lambda_3)^2 - 2(\lambda_1 - \lambda_3)(T_1 - T_3)\right]^{1/2} - (T_1 + T_3)\right\} n_2^2 v^{-1} - v$$

where T_i is defined by (4.9). In the case considered $T_1 = -4$, $T_3 = 2$ and $S^{(2)} = 1.9 \pm 5.0\mathrm{i}$. The plots of the real and the imaginary parts of $S_2^{(2)}(\theta, \varphi) = 1.9 + 5.0\mathrm{i}$ are shown on figure 2(*a*) and figure 2(*b*), respectively.

4.3.3. *The flow* $u_2 = \sin x_1 + \sigma_1 \sin 2x_1 + \sigma_2 \sin x_3$

The conditions obtained in §4.3.1 are not necessary for oscillatory instability to exist. The following simpler flow not satisfying the conditions (4.11) and (4.13) can also give the same type of instability:

$$\left.\begin{aligned} &u_1 = 0, \\ &u_2 = \sin x_1 + \sigma_1 \sin 2x_1 + \sigma_2 \sin x_3, \\ &u_3 = 0, \end{aligned}\right\} \tag{4.19}$$

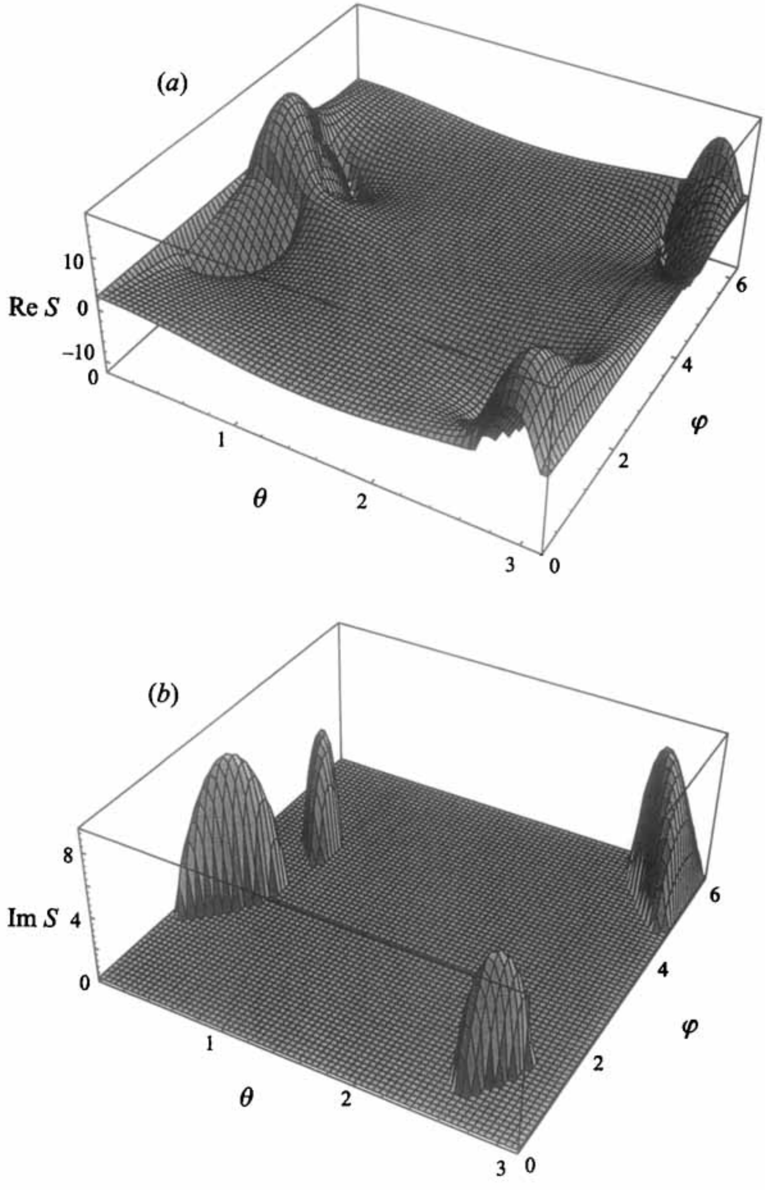

FIGURE 2. (a) The real part and (b) the imaginary part of the growth rate $S_2^{(2)}$.

for example, for $\sigma_1 = \sigma_2 = 2, \nu = 0.1$. The plot of the real and imaginary parts of the growth rate are shown on figure 3(a) and figure 3(b), respectively.

In figure 4 the product of the imaginary and real parts is shown where the latter is positive. The domain where the value of the function is not equal to zero is the oscillatory instability domain. The flow (4.19) is investigated for different values of σ_1 and σ_2. For this flow the parameters λ_i and μ_i, $i = 1, 3$, (4.7) take the following simple form : $\lambda_1 = (4+\sigma_1^2)/8$, $\mu_1 = 3\sigma_1/16$, $\lambda_1 = \sigma_2^2/2$, $\mu_1 = 0$. The maximal value of the real part of the growth rate S in terms of the variables θ, φ for the selected value of the ν is obtained by direct calculations. Varying the value of the ν, one can find a case where $\max[\mathrm{Re}\, S] = 0$. This is the maximum of the marginal stability curve. The growth rate

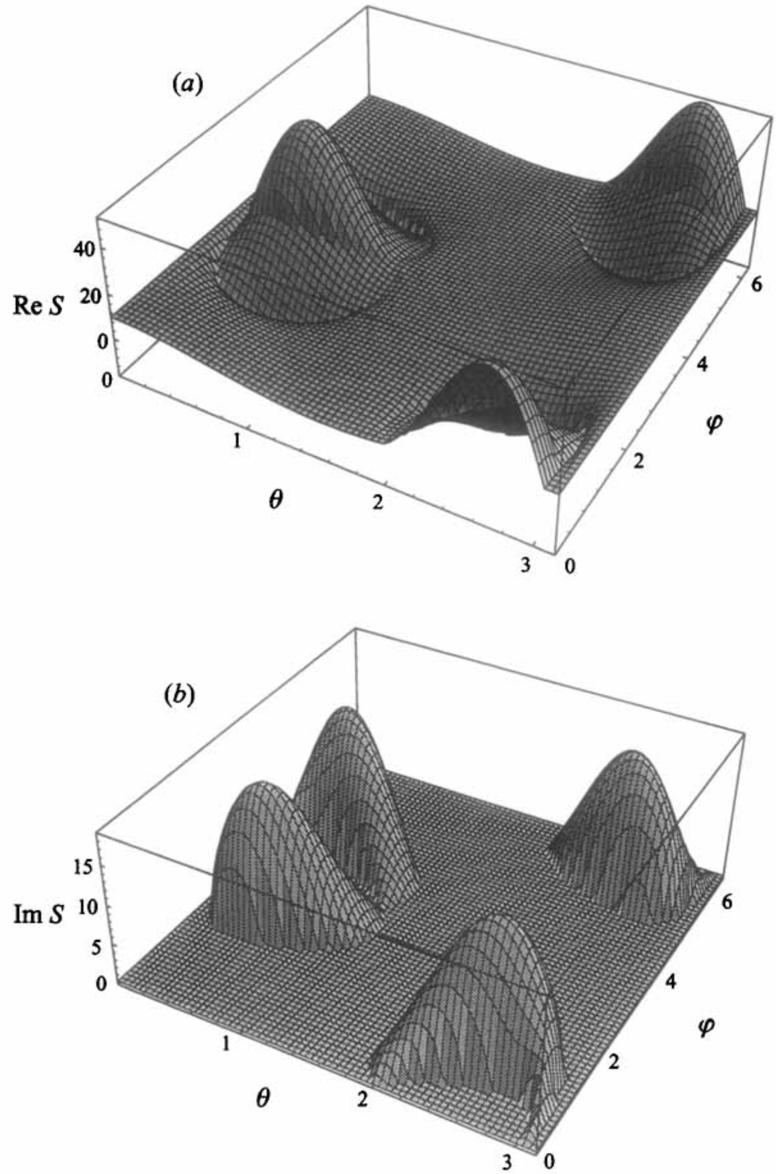

FIGURE 3. (a) The real part and (b) the imaginary part of the growth rate for the flow $u_2 = \sin x_1 + 2\sin 2x_1 + 2\sin x_3$.

at this point ν_{cr} is equal to zero and real only. Thus, the complex growth rate does not correspond to the most dangerous instability. These numerical calculations are done by using the software *Mathematica*. Using this program for the different values of σ_1 and σ_2 we obtained the critical values of the ν. The results are shown in table 1.

5. Concluding remarks

We obtained the general equation (2.40) that determines the long-wave asymptotics of the growth rates. It was found that the instability is connected with the positive

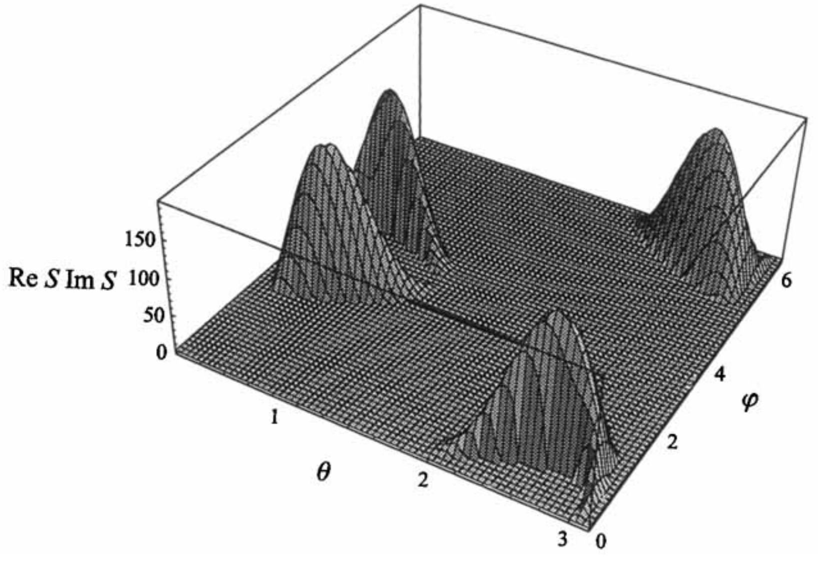

FIGURE 4. The product of the positive real part (we replace the negative real part by zero) and the imaginary part of the growth rate for the flow $u_2 = \sin x_1 + 2\sin 2x_1 + 2\sin x_3$.

σ_2 \ σ_1	0.1	1	1.41	2	10	100
0.1	0.7042	0.7973	0.8817	1.0090	3.6050	35.362
1.0	0.1083	0.5504	0.6733	0.8317	3.5354	35.356
1.41	0.7058	0.6123	0.4998	0.5813	3.4638	35.348
2.0	1.2246	1.1721	1.1178	0.9996	3.3170	35.334
10	7.0353	7.0268	7.0182	7.0000	6.0826	34.648
100	70.707	70.706	70.705	70.704	70.619	61.234

TABLE 1. The critical value of v for the different values of $\sigma_i, i = 1, 2$, in the flow $u_2 = \sin x_1 + \sigma_1 \sin 2x_1 + \sigma_2 \sin x_3$

real part of the coefficient $S^{(2)}$, hence the growth rate is proportional to $|\boldsymbol{\kappa}|^2$. This type of instability is known as 'negative eddy viscosity' (Kraichnan 1976).

Coinciding with the results of Dubrulle & Frisch (1991), we found that the wavevector of the most dangerous disturbances is inclined to the direction of the basic stream for flows of both types (1.1) and (1.2). An unexpected phenomenon has been discovered: an oscillatory instability with the real part of the growth rate proportional to $|\boldsymbol{\kappa}|^2$.

It is necessary to note that the possibility of a complex growth rate ('complex eddy viscosity') in the three-dimensional case is a natural consequence of the non-self-adjointness of the general eigenvalue problem formulated by Dubrulle & Frisch (1991). Recently, some examples of flows with complex eddy viscosity (though with negative $\mathrm{Re}\, S^{(2)}$) were found for some three-dimensional flows (Wirth 1994; Wirth, Gama & Frisch 1995).

In conclusion, let us discuss the nonlinear aspects of the new oscillatory instability. There is a crucial difference between the oscillatory instability of flows periodic in one direction and the oscillatory instability found in the present paper. In the former case, the frequency of long-wave oscillations $\omega \sim k^3$, and the nonlinear regimes are governed by a perturbed Korteveg–de Vries equation (Nepomnyashchy 1995). In the latter case, the frequency of long-wave oscillations $\omega \sim k^2$. A similar situation occurred in a study of oscillatory side-band instabilities in Marangoni convection with deformable interface (Golovin *et al.* 1995). The evolution of amplitudes of interacting waves with different wavenumbers can be described at the leading order of the perturbation theory by a system of Landau equations with cubic interaction terms.

In conclusion, let us emphasize that the Kolmogorov flow and its simplest generalizations turn out to be not quite typical in several aspects, including the absence of a primary oscillatory instability and a specific orientation of the wavevectors of the most dangerous disturbances. These peculiarities may strongly influence the nonlinear evolution of the flow. The extension of the class of flows investigated may clarify the characteristic mechanisms of spontaneous generation of large-scale structures by small-scale flows.

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